An Improved Summation Inequality to Discrete-Time Systems with Time-Varying Delay

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Abstract

The summation inequality plays an important role in developing delay-dependent criteria for discrete-time systems with time-varying delay. This note proposes an improved summation inequality to estimate the summation terms appearing in the forward difference of Lyapunov-Krasovskii functional. Compared with the inequality recently developed by the Wirtinger-based summation inequality and the reciprocally convex lemma, the proposed one reduces the estimation gap while requires the same number of decision variables. A relaxed stability criterion of a linear discrete-time system with a time-varying delay is established by using such novel inequality. Two numerical examples are given to demonstrate the advantages of the proposed method.

Keywords: Discrete-time system, time-varying delay, summation inequality, linear matrix inequality

1. Introduction

In the last few years, the stability analysis of discrete-time systems with time-varying delays has become a hot topic in the field of control theory [1]-[17]. An important objective of stability analysis is to find the admissible delay region such that time-delay systems remain stable for the time-varying delay within this region [18]. Delay-dependent stability criteria developed in the framework of the Lyapunov-Krasovskii functional (LKF) and the linear matrix inequality (LMI) are the most effective criteria to determine such admissible region. The following double summation term is frequently applied during the constructing of LKF to obtain delay-dependent criterion [8]:

$$V_r(k) = \sum_{i=-h_2}^{-h_1-1} \sum_{j=k+i}^{k-1} \eta^T(j) R \eta(j) \tag{1}$$

where h_1 and h_2 are respectively the lower and the upper bounds of a time-varying delay (i.e., $h_1 \leq d(k) \leq h_2$), $R \geq 0$, and $\eta(k) = x(k+1) - x(k)$ with $x(k)$ being the system state. Then the following term will appear in the forward difference of $V_r(k)$:

$$\mathcal{S}(k) := \sum_{i=k-d(k)}^{k-h_1-1} \eta^T(i) R \eta(i) + \sum_{i=k-h_2}^{k-d(k)-1} \eta^T(i) R \eta(i) \tag{2}$$

During the development of stability criteria, a challenging problem is how to estimate the lower bound of the above summation term [9]. Obtaining tighter bound of summation term (i.e., reducing the estimation gap) plays a key role in reducing the conservatism. In the early literature, the free-weighting matrix (FWM) approach [4] and the Jensen-based inequality (JBI) [1] were two important methods for this issue. By relaxing the JBI, Wirtinger-based inequalities (WBIs) were simultaneously reported in [7, 8] and later in [9]. Very recently, an auxiliary function based inequality (AFBI) [10] and a free-matrix-based summation inequality (FMBI) [19] inspired by the research of [22, 23] were developed by further improving the WBI.

Those inequality-based estimation methods include two key steps to estimate $\mathcal{S}(k)$: 1) applying the JBI/WBI/AFBI to estimate two summation terms in $\mathcal{S}(k)$, respectively; and 2) using the reciprocally convex lemma (RCL) [20] to handle the $d(k)$ appearing in the denominator. The recently developed techniques (the WBIs, the AFBI, and the FMIB) focus on the first step. To the best of the authors' knowledge, there is no research that discusses the tighter estimation of $\mathcal{S}(k)$ considering two steps together. This note aims to fill this research gap.

This note proposes an improved summation inequality by considering two terms of $\mathcal{S}(k)$ together. It is tighter than the one obtained by combining the WBI and the RCL but keeps the same number of decision variables. A new stability criterion for a linear discrete-time system with a time-varying delay is established by applying the proposed inequality. Finally, two numerical examples are given to illustrate the effective of the proposed inequality and the corresponding criterion.

Throughout this note, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; $P > 0$ (≥ 0) means that P is a symmetric positive-definite (semi-positive-definite) matrix; diag$\{\cdot\}$ denotes a block-diagonal matrix; Sym$\{X\} = X + X^T$; and the symmetric term in a symmetric matrix is denoted by $*$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem formulation and preliminaries

Consider the following linear discrete-time system with a time-varying delay:

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-d(k)), \ k \geq 0 \\ x(k) = \phi(k), \quad k \in [-h_2, 0] \end{cases} \tag{3}$$

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where $x(k) \in \mathcal{R}^n$ and $\phi(k)$ are the system state and the initial condition, respectively; A and A_d are the system matrices; and $d(k)$ is a positive integer which is time-varying and satisfies

$$1 \le h_1 \le d(k) \le h_2 \tag{4}$$

This note is concerned with the stability of system (3). As mentioned in Section I, a challenging problem is how to estimate the summation term $\mathcal{S}(k)$. Therefore, the first aim of this note is to develop a more effective estimation method for this task. Then, this note will apply the proposed method to derive a new stability criterion for judging the influence of the time-varying delay on the stability of system.

Several WBIs with different forms were simultaneously reported in [7, 8] and later in [9]. The ones to be applied in this note are recalled from [8].

Lemma 1. *(Wirtinger-based inequality [8]) For a given symmetric positive definite matrix R, integers $b > a$, any sequence of discrete-time variable x: $\mathcal{Z}[a,b] \rightarrow \mathcal{R}^n$, the following inequalities hold*

$$\sum_{i=a}^{b-1} \eta^T(i)R\eta(i) \ge \frac{1}{b-a}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}^T \begin{bmatrix} R & 0\\ 0 & 3\left(\frac{b-a+1}{b-a-1}\right)R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} \tag{5}$$

$$\ge \frac{1}{b-a}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}^T \begin{bmatrix} R & 0\\ 0 & 3R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} \tag{6}$$

where $\vartheta_1 = x(b) - x(a)$, and $\vartheta_2 = x(b) + x(a) - 2\sum_{i=a}^{b} \frac{x(i)}{b-a+1}$.

The following RCL is usually applied to combine with the WBI in the literature.

Lemma 2. *(Reciprocally convex lemma (RCL) [20]) For a given scalar α in the interval $(0, 1)$, symmetric positive definite matrices U_1 and U_2, and any matrix X such that $\begin{bmatrix} U_1 & X\\ * & U_2\end{bmatrix} \ge 0$, the following inequality holds*

$$\begin{bmatrix} \frac{1}{\alpha}U_1 & 0\\ * & \frac{1}{1-\alpha}U_2\end{bmatrix} \ge \begin{bmatrix} U_1 & X\\ * & U_2\end{bmatrix} \tag{7}$$

The estimation of the $\mathcal{S}(k)$ via the WBI and the RCL leads to the following lemma.

Lemma 3. *For a symmetric positive definite matrix R, any matrix X satisfying $\begin{bmatrix} \tilde{R} & X\\ * & \tilde{R}\end{bmatrix} \ge 0$ with $\tilde{R} = \mathrm{diag}\{R, 3R\}$, the $\mathcal{S}(k)$ defined in (2) is estimated as*

$$\mathcal{S}(k) \ge \frac{1}{h_{21}}\zeta^T(k)\begin{bmatrix}E_1\\ E_2\end{bmatrix}^T \begin{bmatrix} \tilde{R} & X\\ * & \tilde{R}\end{bmatrix}\begin{bmatrix}E_1\\ E_2\end{bmatrix}\zeta(k) \tag{8}$$

where

$$h_{21} = h_2 - h_1, \quad d = d(k) \tag{9}$$

$$\zeta(k) = \Big[x^T(k), x^T(k-h_1), x^T(k-d), x^T(k-h_2), v_1^T(k), v_2^T(k), v_3^T(k)\Big]^T \tag{10}$$

$$v_1(k) = \sum_{i=k-h_1}^{k} \frac{x(i)}{h_1+1}, \qquad v_2(k) = \sum_{i=k-d}^{k-h_1} \frac{x(i)}{d-h_1+1} \tag{11}$$

$$v_3(k) = \sum_{i=k-h_2}^{k-d} \frac{x(i)}{h_2-d+1} \tag{12}$$

$$E_1 = \begin{bmatrix} e_2 - e_3\\ e_2 + e_3 - 2e_6\end{bmatrix}, \quad E_2 = \begin{bmatrix} e_3 - e_4\\ e_3 + e_4 - 2e_7\end{bmatrix} \tag{13}$$

$$e_i = \left[0_{n\times(i-1)n}, I_{n\times n}, 0_{n\times(7-i)n}\right], i = 1, 2, \cdots, 7 \tag{14}$$

Proof: Using WBI (6) and RCL (7) to estimate the $\mathcal{S}(k)$ yields

$$\begin{aligned}\mathcal{S}(k) &= \sum_{i=k-d}^{k-h_1-1} \eta^T(i)R\eta(i) + \sum_{i=k-h_2}^{k-d-1} \eta^T(i)R\eta(i)\\ &\ge \frac{1}{d-h_1}\zeta^T(k)E_1^T\tilde{R}E_1\zeta(k) + \frac{1}{h_2-d}\zeta^T(k)E_2^T\tilde{R}E_2\zeta(k)\\ &\ge \frac{1}{h_{21}}\zeta^T(k)\begin{bmatrix}E_1\\ E_2\end{bmatrix}^T \begin{bmatrix} \tilde{R} & X\\ * & \tilde{R}\end{bmatrix}\begin{bmatrix}E_1\\ E_2\end{bmatrix}\zeta(k)\end{aligned}$$

3. A relaxed summation inequality

This section develops an improved summation inequality for estimating $\mathcal{S}(k)$, shown in the following lemma.

Lemma 4. *For a symmetric positive definite matrix R, any matrix X, the $\mathcal{S}(k)$ defined in (2) is estimated as*

$$\mathcal{S}(k) \ge \frac{1}{h_{21}}\zeta^T(k)\begin{bmatrix}E_1\\ E_2\end{bmatrix}^T \left(\begin{bmatrix} \tilde{R} & X\\ * & \tilde{R}\end{bmatrix} + \begin{bmatrix} \frac{h_2-d}{h_{21}}T_1 & 0\\ 0 & \frac{d-h_1}{h_{21}}T_2\end{bmatrix}\right)\begin{bmatrix}E_1\\ E_2\end{bmatrix}\zeta(k) \tag{15}$$

where $\tilde{R} = \mathrm{diag}\{R, 3R\}$, $T_1 = \tilde{R} - X\tilde{R}^{-1}X^T$ and $T_2 = \tilde{R} - X^T\tilde{R}^{-1}X$.

Proof: Firstly, for the symmetric matrix $R > 0$ and any matrices, $M_i, i = 1, 2, 3, 4$, with appropriate dimension, the following holds based on Schur complement:

$$\begin{bmatrix} \begin{bmatrix} M_{2i-1}\\ M_{2i}\end{bmatrix} R^{-1} \begin{bmatrix} M_{2i-1}\\ M_{2i}\end{bmatrix}^T & \begin{bmatrix} M_{2i-1}\\ M_{2i}\end{bmatrix}\\ * & R\end{bmatrix} \ge 0, \ i = 1, 2$$

Then, for any vector $\varsigma_j(k, i)$, $j = 1, 2$, the following is true:

$$\Pi_1 = \sum_{i=k-d}^{k-h_1-1} \varsigma_1^T(k,i)\begin{bmatrix} M_1R^{-1}M_1^T & M_1R^{-1}M_2^T & M_1\\ * & M_2R^{-1}M_2^T & M_2\\ * & * & R\end{bmatrix}\varsigma_1(k,i) \ge 0 \tag{16}$$

$$\Pi_2 = \sum_{i=k-h_2}^{k-d-1} \varsigma_2^T(k,i)\begin{bmatrix} M_3R^{-1}M_3^T & M_3R^{-1}M_4^T & M_3\\ * & M_4R^{-1}M_4^T & M_4\\ * & * & R\end{bmatrix}\varsigma_2(k,i) \ge 0 \tag{17}$$

Secondly, letting $f(i, a, b) = \frac{2i-b-a+1}{b-a+1}$ yields the following equalities

$$\sum_{i=k-d}^{k-h_1-1}\eta(i) = (e_2 - e_3)\zeta(k), \quad \sum_{i=k-d}^{k-h_1-1} f_1(i)\eta(i) = (e_2 + e_3 - 2e_6)\zeta(k) \tag{18}$$

$$\sum_{i=k-h_2}^{k-d-1}\eta(i) = (e_3 - e_4)\zeta(k), \quad \sum_{i=k-h_2}^{k-d-1} f_2(i)\eta(i) = (e_3 + e_4 - 2e_7)\zeta(k) \tag{19}$$

$$\sum_{i=k-d}^{k-h_1-1} 1 = d-h_1,\ \sum_{i=k-d}^{k-h_1-1} f_1(i)=0,\ \sum_{i=k-d}^{k-h_1-1} f_1^2(i)=\frac{(d-h_1)(d-h_1-1)}{3(d-h_1+1)} \quad (20)$$

$$\sum_{i=k-h_2}^{k-d-1} 1 = h_2-d,\ \sum_{i=k-h_2}^{k-d-1} f_2(i)=0,\ \sum_{i=k-h_2}^{k-d-1} f_2^2(i)=\frac{(h_2-d)(h_2-d-1)}{3(h_2-d+1)} \quad (21)$$

where $f_1(i) = f(i, k-d, k-h_1)$ and $f_2(i) = f(i, k-h_2, k-d)$. Moreover, redefine vector $\varsigma_j(k,i)$, $j = 1, 2$ and matrices $M_i, i = 1, 2, 3, 4$, in Π_i as follows

$$\varsigma_j(k,i) = \left[g^T(k),\ f_j(i)g^T(k), \eta^T(i)\right]^T,\ g(k) = [E_1^T,\ E_2^T]^T\zeta(k) \quad (22)$$

$$M_1 = -\frac{1}{h_{21}}\left[R,\ 0,\ L_1^T\right]^T, \quad M_2 = -\frac{1}{h_{21}}\left[0,\ 3R,\ L_2^T\right]^T \quad (23)$$

$$M_3 = -\frac{1}{h_{21}}\left[L_3^T,\ R,\ 0\right]^T, \quad M_4 = -\frac{1}{h_{21}}\left[L_4^T,\ 0,\ 3R\right]^T \quad (24)$$

$$X = [L_1, L_2]^T = [L_3, L_4],\quad \tilde{R} = \mathrm{diag}\{R, 3R\} \quad (25)$$

where $L_i, i = 1, 2, 3, 4$ are any matrices.

Thirdly, calculating $\Pi_i, i = 1, 2$ based on (18)-(25). It follows from (20), (22), (23), (25) and $\frac{d-h_1-1}{d-h_1+1} < 1$ that

$$\begin{aligned}
&\sum_{i=k-d}^{k-h_1-1}\begin{bmatrix} g(k) \\ f_1(i)g(k)\end{bmatrix}^T \begin{bmatrix} M_1R^{-1}M_1^T & M_1R^{-1}M_2^T \\ * & M_2R^{-1}M_2^T\end{bmatrix}\begin{bmatrix} g(k) \\ f_1(i)g(k)\end{bmatrix} \\
&= (d-h_1)g^T(k)M_1R^{-1}M_1^Tg(k) + 2\times 0\times g^T(k)M_1R^{-1}M_2^Tg(k) \\
&\quad + \frac{(d-h_1)(d-h_1-1)}{3(d-h_1+1)}g^T(k)M_2R^{-1}M_2^Tg(k) \\
&< (d-h_1)g^T(k)M_1R^{-1}M_1^Tg(k) + (d-h_1)g^T(k)M_2(3R)^{-1}M_2^Tg(k) \\
&= \frac{d-h_1}{h_{21}^2}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T \left[\begin{array}{cc|c} R & 0 & L_1^T \\ 0 & 0 & 0 \\ \hline L_1 & 0 & L_1R^{-1}L_1^T\end{array}\right]\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k) \\
&\quad + \frac{d-h_1}{h_{21}^2}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 3R & L_2^T \\ \hline 0 & L_2 & L_2(3R)^{-1}L_2^T\end{array}\right]\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k) \\
&= \frac{d-h_1}{h_{21}^2}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T \left[\begin{array}{cc|c} R & 0 & L_1^T \\ 0 & 3R & L_2^T \\ \hline L_1 & L_2 & L_1R^{-1}L_1^T + L_2(3R)^{-1}L_2^T\end{array}\right]\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k) \\
&= \frac{d-h_1}{h_{21}^2}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T \begin{bmatrix} \tilde{R} & X \\ * & X^T\tilde{R}^{-1}X\end{bmatrix}\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k) \qquad (26)
\end{aligned}$$

It follows from (18), (22), (23), (25) that

$$\begin{aligned}
&\sum_{i=k-d}^{k-h_1-1}\left\{\begin{bmatrix} g(k) \\ f_1(i)g(k)\end{bmatrix}^T\begin{bmatrix} M_1 \\ M_2\end{bmatrix}\eta(i) + \eta^T(i)\begin{bmatrix} M_1 \\ M_2\end{bmatrix}^T\begin{bmatrix} g(k) \\ f_1(i)g(k)\end{bmatrix}\right\} \\
&= 2\sum_{i=k-d}^{k-h_1-1}\left\{g^T(k)M_1\eta(i) + f_1(i)g^T(k)M_2\eta(i)\right\} \\
&= 2g^T(k)\begin{bmatrix} M_1 & M_2\end{bmatrix}\begin{bmatrix} \sum_{i=k-d}^{k-h_1-1}\eta(i) \\ \sum_{i=k-d}^{k-h_1-1}f_1(i)\eta(i)\end{bmatrix} \\
&= -\frac{2}{h_{21}}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T\begin{bmatrix} R & 0 \\ 0 & 3R \\ L_1 & L_2\end{bmatrix}E_1\zeta(k) \\
&= -\frac{1}{h_{21}}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T\left[\begin{array}{cc|c} 2R & 0 & L_1^T \\ 0 & 6R & L_2^T \\ \hline L_1 & L_2 & 0\end{array}\right]\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k) \\
&= -\frac{1}{h_{21}}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T\begin{bmatrix} 2\tilde{R} & X \\ * & 0\end{bmatrix}\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k) \qquad (27)
\end{aligned}$$

Similarly, based on (19), (21), (22), (24), (25), and $\frac{h_2-d-1}{h_2-d+1} < 1$, the following holds

$$\begin{aligned}
&\sum_{i=k-h_2}^{k-d-1}\begin{bmatrix} g(k) \\ f_2(i)g(k)\end{bmatrix}^T\begin{bmatrix} M_3R^{-1}M_3^T & M_3R^{-1}M_4^T \\ * & M_4R^{-1}M_4^T\end{bmatrix}\begin{bmatrix} g(k) \\ f_2(i)g(k)\end{bmatrix} \\
&< \frac{h_2-d}{h_{21}^2}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T\left[\begin{array}{c|cc} L_3R^{-1}L_3^T + L_4(3R)^{-1}L_4^T & L_3 & L_4 \\ \hline L_3^T & R & 0 \\ L_4^T & 0 & 3R\end{array}\right]\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k) \\
&= \frac{h_2-d}{h_{21}^2}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T\begin{bmatrix} X\tilde{R}^{-1}X^T & X \\ * & \tilde{R}\end{bmatrix}\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k) \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=k-h_2}^{k-d-1}\left\{\begin{bmatrix} g(k) \\ f_2(i)g(k)\end{bmatrix}^T\begin{bmatrix} M_3 \\ M_4\end{bmatrix}\eta(i) + \eta^T(i)\begin{bmatrix} M_3 \\ M_4\end{bmatrix}^T\begin{bmatrix} g(k) \\ f_2(i)g(k)\end{bmatrix}\right\} \\
&= -\frac{1}{h_{21}}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T\left[\begin{array}{c|cc} 0 & L_3 & L_4 \\ \hline L_3^T & 2R & 0 \\ L_4^T & 0 & 6R\end{array}\right]\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k) \\
&= -\frac{1}{h_{21}}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T\begin{bmatrix} 0 & X \\ * & 2\tilde{R}\end{bmatrix}\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k) \qquad (29)
\end{aligned}$$

Thus, combining (16), (17), and (26)-(29) yields

$$\begin{aligned}
&\Pi_1 + \Pi_2 \qquad (30) \\
&< \mathcal{S}(k) - \frac{1}{h_{21}}\zeta^T(k)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}^T\left(\begin{bmatrix} \tilde{R} & X \\ * & \tilde{R}\end{bmatrix} + \begin{bmatrix} \frac{h_2-d}{h_{21}}T_1 & 0 \\ 0 & \frac{d-h_1}{h_{21}}T_2\end{bmatrix}\right)\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\zeta(k)
\end{aligned}$$

Finally, the summation inequality (15) can be obtained based on $\Pi_1 + \Pi_2 \geq 0$ and (30). This completes the proof. ■

Remark 1. *On the one hand, it is obvious that $T_i \geq 0, i = 1, 2$ hold during slack matrix X is selected to satisfy $\begin{bmatrix} \tilde{R} & X \\ * & \tilde{R}\end{bmatrix} \geq 0$ (it is the requirement of Lemma 3). Thus, T_i-dependent positive term appearing in (15) reduces the estimation gap between two sides of (8). Thus, the proposed inequality (15) is tighter than (8). On the other hand, the slack matrix introduced by inequality (15) (i.e., X) is the same as the one arising in (8). That is to say, compared with inequality (8), the proposed (15) is a variable-increase-free inequality. Therefore, inequality (15) has the potential to derive new criteria that have less conservatism but require the same number of decision variables.*

Remark 2. *The AFBI proposed in [10] improves the WBI by adding additional terms to reduce the estimation gap existing in the WBI. The FMBI proposed in [19] improves the WBI by introducing many free matrices. Both improvements target to a single summation term, i.e., the first step for handling $\mathcal{S}(k)$ as mentioned in Section I. The idea of deriving inequality (15) provides a new way to improve the WBI, i.e., considering two summation terms of $\mathcal{S}(k)$ together.*

Remark 3. *The techniques for continuous time-delay system are usually similar to the ones for discrete-time systems with time-varying delay (for example, Wirtinger-based integral inequality for continuous-time systems and Wirtinger-based summation inequality for discrete-time systems). It is expected that the corresponding new integral inequalities for continuous-time systems would be developed based on the similar idea of deriving of summation inequality (15).*

4. A novel stability criterion

By using the summation inequality (15) and the LKF taken from [8], the following stability criterion for system (3) is established.

Theorem 1. *For given integers h_1 and h_2, system (3) with a time-varying delay satisfying (4) is asymptotically stable if there exist symmetric positive definite matrices $P \in \mathcal{R}^{3n\times 3n}$, $Q_1 \in \mathcal{R}^{n\times n}$, $Q_2 \in \mathcal{R}^{n\times n}$, $R_1 \in \mathcal{R}^{n\times n}$, $R \in \mathcal{R}^{n\times n}$, and any matrix $X \in \mathcal{R}^{2n\times 2n}$, such that the following LMIs hold:*

$$\begin{bmatrix} \Psi(h_1) - \Upsilon_{5,1} & E_1^T X \\ * & -\tilde{R} \end{bmatrix} < 0 \tag{31}$$

$$\begin{bmatrix} \Psi(h_2) - \Upsilon_{5,2} & E_2^T X^T \\ * & -\tilde{R} \end{bmatrix} < 0 \tag{32}$$

where

$$\begin{aligned}
\Psi(d) &= \Upsilon_1(d) + \Upsilon_2 + \Upsilon_3 - \Upsilon_4 \\
\Upsilon_1(d) &= \mathrm{Sym}\{\Gamma^T(d)P(\Gamma_1 - \Gamma_2)\} + \Gamma_1^T P\Gamma_1 - \Gamma_2^T P\Gamma_2 \\
\Gamma(d) &= \left[e_1^T,\ (h_1+1)e_5^T,\ (d-h_1+1)e_6^T + (h_2-d+1)e_7^T\right]^T \\
\Gamma_1 &= \left[e_s^T,\ -e_2^T,\ -e_3^T - e_4^T\right]^T \\
\Gamma_2 &= \left[e_0^T,\ -e_1^T,\ -e_2^T - e_3^T\right]^T \\
\Upsilon_2 &= e_1^T Q_1 e_1 - e_2^T Q_1 e_2 + e_2^T Q_2 e_2 - e_4^T Q_2 e_4 \\
\Upsilon_3 &= e_s^T(h_1^2 R_1 + h_{21}^2 R)e_s \\
\Upsilon_4 &= \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_5 \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & 3\left(\frac{h_1+1}{h_1-1}\right)R_1 \end{bmatrix} \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_5 \end{bmatrix}
\end{aligned} \tag{33}$$

$$\begin{aligned}
\Upsilon_{5,1} &= \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} 2\tilde{R} & X \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \\
\Upsilon_{5,2} &= \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & X \\ * & 2\tilde{R} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \\
E_1 &= \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_6 \end{bmatrix}, \quad E_2 = \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_7 \end{bmatrix} \\
e_0 &= 0_{7n\times 7n}, \quad e_s = (A - I)e_1 + A_d e_3 \\
e_i &= \left[0_{n\times(i-1)n}, I_{n\times n}, 0_{n\times(7-i)n}\right], i = 1, 2, \cdots, 7
\end{aligned}$$

Proof: Consider the LKF candidate taken from [8]:

$$\begin{aligned}
V(x_k) &= \xi^T(k)P\xi(k) + \sum_{i=k-h_1}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-h_2}^{k-h_1-1} x^T(i)Q_2x(i) \\
&\quad + h_1\sum_{i=-h_1}^{-1}\sum_{j=k+i}^{k-1}\eta^T(j)R_1\eta(j) + h_{21}\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1}\eta^T(j)R\eta(j)
\end{aligned} \tag{34}$$

where $P > 0$, $Q_i > 0$, $i = 1, 2$, $R_1 > 0$, $R > 0$, and

$$\xi(k) = \left[x^T(k), \sum_{i=k-h_1}^{k-1} x^T(i), \sum_{i=k-h_2}^{k-h_1-1} x^T(i)\right]^T, \quad \eta(k) = x(k+1) - x(k)$$

Calculating the forward difference of $V(x_k)$ yields [8]:

$$\begin{aligned}
\Delta V(x_k) &= \zeta^T(k)[\Upsilon_1(d) + \Upsilon_2 + \Upsilon_3]\zeta(k) \\
&\quad - h_1\sum_{i=k-h_1}^{k-1}\eta^T(i)R_1\eta(i) - h_{21}\sum_{i=k-h_2}^{k-h_1-1}\eta^T(i)R\eta(i)
\end{aligned} \tag{35}$$

where $\zeta(k)$ is defined in (10).

Using WBI (5) to estimate R_1-dependent summation term yields

$$h_1\sum_{i=k-h_1}^{k-1}\eta^T(i)R_1\eta(i) \geq \zeta^T(k)\Upsilon_4\zeta(k) \tag{36}$$

And using the proposed summation inequality (15) to estimate R-dependent summation term $(= h_{21}\mathcal{S}(k))$ yields

$$h_{21}\sum_{i=k-h_2}^{k-h_1-1}\eta^T(i)R\eta(i) \geq \zeta^T(k)\tilde{\Upsilon}_5(d)\zeta(k) \tag{37}$$

where

$$\tilde{\Upsilon}_5(d) = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \left(\begin{bmatrix} \tilde{R} & X \\ * & \tilde{R} \end{bmatrix} + \begin{bmatrix} \frac{h_2-d}{h_{21}}T_1 & 0 \\ 0 & \frac{d-h_1}{h_{21}}T_2 \end{bmatrix}\right) \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

Thus, based on (35)-(37), the forward difference of the LKF is estimated as

$$\begin{aligned}
\Delta V(x_k) &\leq \zeta^T(k)[\Upsilon_1(d) + \Upsilon_2 + \Upsilon_3 - \Upsilon_4 - \tilde{\Upsilon}_5(d)]\zeta(k) \\
&:= \zeta^T(k)\Phi(d)\zeta(k)
\end{aligned} \tag{38}$$

It is easy to check that $\Phi(d)$ is affine with respect to the time-varying delay $d \in [h_1, h_2]$, thus $\Phi(d) < 0$ if and only if $\Phi(h_1) < 0$ and $\Phi(h_2) < 0$, which are equivalent to LMIs (31) and (32), respectively, based on Schur complement. Therefore, if LMIs (31) and (32) hold, then $\Delta V(x_k) \leq -\varepsilon\|x(k)\|^2$ for a sufficient small $\varepsilon > 0$, which shows the asymptotical stability of system (3). This completes the proof. ■

Remark 4. *In [8], the summation term, $\mathcal{S}(k)$, was estimated by using the WBI and the RCL, while it is handled by using a tighter inequality (15) in this note. On the other side, the matrices to be determined in Theorem 1 are the same as the ones in the criterion in [8] (Theorem 5 therein). Thus, Theorem 1 has the potential to provide less conservative results while requires the same number of decision variables in comparison to the one in [8].*

Remark 5. *In [10], the AFBI tighter than the WBI, together with the RCL, was applied to improve the criterion of [8], and in [19], the FMBI including the WBI was used to improve the WBI-based criterion [8]. Both improved criteria require the increase of the number of decision variables. On the contrary, Theorem 1 improves the criterion in [8] but does not require additional decision variables.*

Remark 6. *Theorem 1 can be further improved by combining the proposed inequality with several existing techniques, such as introducing zero-value terms for estimating the forward difference of the LKF [12], and constructing augmented-based and/or delay-partition-based LKF [7, 10]. The details are omitted in this note, since the main contribution of this note is to propose a novel way to improve the WBI while keep the same number of decision variables, and there is no much technique difficulty in the aforementioned extension.*

5. Numerical examples

Two numerical examples are used to demonstrate the advantages of the proposed criterion via the comparisons of the calculated maximal admissible delay upper bounds (MAUBs) and of the number of decision variables (NDVs).

Example 1. *Consider system (3) with*

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix} \tag{39}$$

This example is widely used for checking the conservatism of stability criteria. The MAUBs calculated by Theorem 1, together with the ones reported in several literature, are given in Table 1, where the NDVs of several criteria are also listed. The following observations are summarized from the results listed in table:

- In the early literature, stability criteria were developed via the FWM approach [1, 2, 3, 4, 5] or the JBI [13, 15, 11], commonly combined with a simple LKF [1, 2, 3, 4, 15], and they are very conservatism. Then, improved criteria were established by constructing LKFs with more general form (delay-partition-based LKF [6] and augmented-based LKF [12]) and/or by replacing the JBI with tighter WBI [7, 8]. The WBI reduces the conservatism and avoids much increase of NDVs. (Note that Theorem 4 of [6] requires tune three parameters $d_i, i = 1, 2, 3$, which is a time-consuming procedure, although its NOV is smaller than that of the WBI-based criteria in [7, 8].)
- Very recently, two types of methods were developed to improve the WBI, including the AFBI and the FMBI. It can be found from the table that both the AFBI-based criterion [10] and the FMBI-based criterion [19] achieve the reduction of conservatism at the cost of increase of NDVs. On the contrary, Theorem 1 obtained by the proposed inequality provides less conservative results but keeps the same NDVs in comparison with the WBI-based criterion [8, 7]. Moreover, Theorem 1 provides better results than the AFBI- and FMBI-based criteria [10, 19] but requires smaller NDVs. It clearly shows the advantages of the proposed inequality and the corresponding criterion.

Example 2. *Consider system (3) with*

$$A = \begin{bmatrix} 0.6480 & 0.0400 \\ 0.1200 & 0.6540 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1512 & -0.0518 \\ 0.0259 & -0.1091 \end{bmatrix} \tag{40}$$

Table 1: The MAUBs and the NDVs for different criteria (Example 1)

Methods	h_1					NDVs ($n=2$)
	2	4	6	9	11	
[1-5,11,13,15]	<20	<20	≤20	≤21	<22	
[6] (Theorem $4_{l=3}$)	21	21	21	22	23	$7.5n^2+4.5n$
[12] (Theorem 2)	22	22	22	22	23	$27n^2+9n$
[8] (Theorem 5)	20	21	21	22	23	$10.5n^2+3.5n$
[7] (Remark 4)	20	21	21	22	23	$10.5n^2+3.5n$
[10] (Remark 6)	20	21	21	22	23	$20.5n^2+5.5n$
[10] (Theorem 1)	20	21	21	22	23	$29.5n^2+8.5n$
[19] (Theorem 1)	21	22	22	23	23	$78.5n^2+12.5n$
Theorem 1	21	22	22	23	24	$10.5n^2+3.5n$

This example is recalled from the recently published literature [19]. As mentioned in Remarks 1 and 2, the proposed inequality and both the AFBI and the FMBI can be considered as improvements of the WBI. In this example, Theorem 1 proposed in this note is compared with the criteria obtained through the WBI [8], the AFBI [10], and the FMBI [19]. The MAUBs calculated by those criteria, together with the corresponding NDVs, are summarized in Table 2. Theorem 1 provides less conservative results but keeps the same NDVs in comparison with the WBI-based criterion [8], and it leads to better results than the FMBI-based criterion [19] and the AFBI-based criterion [10] with requiring smaller NDVs. It means that Theorem 1 has the potential to reduce the conservatism of the WBI-based criteria without introducing much extra decision variables.

Table 2: The MAUBs and the NDVs for different criteria (Example 2)

Methods	h_1					NDVs ($n=2$)
	5	7	11	13	20	
[8] (Theorem 5)	20	22	25	27	34	$10.5n^2+3.5n$
[10] (Theorem 1)	20	22	26	28	34	$29.5n^2+8.5n$
[19] (Theorem 1)	21	22	26	27	34	$78.5n^2+12.5n$
Theorem 1	21	22	26	28	35	$10.5n^2+3.5n$

6. Conclusions

This note has proposed a novel summation inequality by considering two summation terms appearing in the forward difference of the LKF together. Compared with the recently reported inequality derived by the WBI and the RCL, the proposed one reduces the estimation gap while requires the same NDVs. It is a new way to reduce the conservatism caused by the inequality based estimation. Application this inequality to the linear discrete-time system with a time-varying delay has lead to a relaxed stability criterion. Two numerical examples have been given to demonstrate the advantages of the proposed method.

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